The Dozenal Society of America

Trigons to Triads

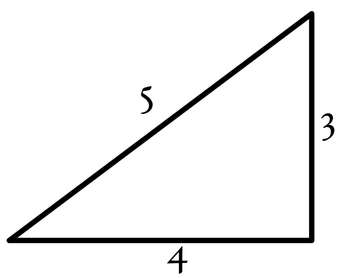
A Play on Chords

Troy

This is the story of a device. Simple to make and use, it nevertheless quite profoundly demonstrates another felicitous and unique feature of base twelve. There certainly could not be a decimal version.

* * *

The primal importance of *three* and the somewhat mysterious, borderline significance of *five* having been surveyed at length in previous essays (Journals 6 and 8), it now seems appropriate to consider a phenomenon involving both of these numbers. Mentioning that the said happenstance also involves *two* — in the guise of its second power, four — gives the starting-point of this article, which is the 3,4,5 triangle.



When I first learned of this unique trigon at school, I was astonished that anything could be so neat: having learned Pythagoras' Theorem (and how to prove it, using a very cumbersome method which took a page-and-a-half of the exercise book), we of Form IV had become resigned to being given two sides of a right-angled triangle and finding the third side to be a surd (needing messy decimal approximation to find actual length). We all got used to $1,\sqrt{3},2$ or $1,2,\sqrt{5}$ or $1,1,\sqrt{2}$ and so on; hence, to be shown a case involving not only three simple integers but also *consecutive* integers pleased us greatly (it must have seemed more like a gift from the gods to the ancient Sumerians and Egyptians).

A brief flurry of experiments convinced us — although we could not prove it at the time — that this was indeed a unique case: $3^2+4^2 = 5^2$ was alone. Although there is an infinity of integral Pythagorean triples (see Mathematical Section for an easy way of finding them), such as 5,10,11, etc., there are no other *consecutive* sets, and even at that tender age some of us felt that it must mean something: that the 3,4,5 set was perhaps one of those tiny windows in the general opacity of things; a little bright region (like finding that the sine of one-third of a right-angle is one-half, for example) through which Nature might be glimpsed.

The 3,4,5 triangle has been used by builders throughout history to make right-angles (and to give easy-to-measure roof profiles), and it was undoubtedly this particular trigon which inspired deduction of the general law named after Pythagoras.[1] But is there anything else that these numbers mean?

There are three rigid space-frames comprising equilateral triangles only; these have three, four, and five vertices respectively.

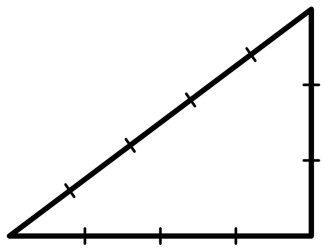
The regular plane figures which comprise the faces of regular solids are the equilateral triangle, the square, and the regular pentagon: 3, 4, and 5 sides. There are no others.

$3 \times 4 \times 5$ equals sixty, or five dozen. Did this have any bearing on the Sumerians' choice of sixty as a secondary number-base, which we still use today for time and angle? It is an ideal number for the job. It is, too, worth noting that resolution of sixty into prime factors puts the four first: $2^2 \times 3 \times 5$.

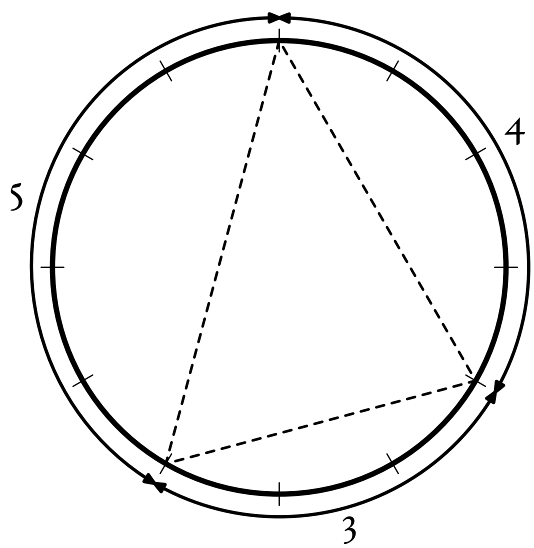
The property wnich beckoned me most strongly, however, was the sum: $3 + 4 + 5 =$ *1 dozen*.

It is, surely, not too radical to imagine an old-world foreman (taskmaster?) carrying a light cord twelve cubits long, graduated with ink or dye spots, which would serve not only to check distances and heights but also — with the help of fine wood or metal pins — to set-out right-angles where needed? The twelve-cubit line would make the *perimeter* of the triangle whose vertices would thus divide the dozen units in the *ratio* 3 : 4 : 5.

[1] There is evidence that the Babylonians knew about triple a,b,c where $a^2 + b^2 = c^2$, *and* could calculate square roots using their sexagesimal base of numeration.



Marking these vertex-points and laying-out the imagined cord in a circle gave me a diagram for Modulo twelve — very much the same sort of thing as a clock-dial — divided in the same ratio.



Joining the ratio-points with chords gave another triangle (dashed lines). The dial-scale thus became the circumscribed circle of this new trigon, which could then be imagined rotating about the centre like a three-way pointer, indicating the 3 : 4 : 5 (or, in this diagram, 4 : 3 : 5) groups of digits for Mod. twelve (0470, 2692, etc.).

Bells now rang. The pattern looked familiar. What else, apart from clock-dials, went in Mod. twelve? Of course! The music scale of a dozen semi-tones: the "well-tempered" scale (thoroughly described by Dr. Impagliazzo of the DSA recently, and the occurrence of which in church-organ pipe-numbering first stirred duodecimal thoughts in the late Tom Pendlehury's mind) of the piano is just such a Modulo-twelve cycle.

A chromatic scale, starting with C in the top (zero) position, was duly written clockwise round the dial. Lo and behold! The vertices of the little triangle now pointed to C-E-G-C, the chord of C major. Moving the trigon round one-sixth of a turn gave D-F♯-A-D, or D major, and so on. *This was nothing less than a rational chord-finder*, courtesy of base twelve and that unique 3,4,5 triple: simply positioning the appropriate vertex of the trigon at a given note automatically selected the other two motes needed for the major triad in that key. One could, literally, dial a chord.

There was more. Turning the rotating triangle over gave such combinations as C-E♭-G-C and G-B♭-D-G; in other words, the minor triads. Wonderful! (I suppose we could say that G minor is a *reflection* of D major? They are musically related.)

* * *

The device was simple to make: a baseboard of ¼″ plywood (about A4 size: 8¼″ by ɛ⅔″) took a paper diagram, pasted on, and the rotating triangle was cut from ⅛″ hardboard with an axle made from ¼″ dowelling long enough to protrude *each side* about ¼″ (to enable it to be turned-over). The "major chord" face of the triangular pointer was painted red, the "minor chord" face blue, with each keynote vertex marked by a gold dot.

* * *

So, what have we here? An amusing guide for canteen-pianists, guitar-players, and amateur barber-shop quartets, perhaps the chord-finder might well also be of real, constructive use for children — in school or at home — in these days when musical education places much emphasis on the composition of simple tunes using electronic keyboards. Certainly, it illustrates yet another fundamental application of the dozen (as remarked earlier, no *decimal* version is possible); and representatives of Dozenal Societies might do well to have the device handy for such media interviews as occasionally come our way.

If any member, having decided to make one of these, would care to inveigle the music-teacher of the local school into trying it out with children, I should be most grateful for a brief report on its success or failure as a learning aid.

Finding Pythagorean Triples

Taking any two numbers, p and q, then wrting expressions for the difference and sum of their squares, we get:

$$p^2 - q^2 \qquad p^2 + q^2 \tag{1}$$

Squaring these expressions and comparing, we have:

$$(p^2 - q^2)^2 \qquad (p^2 + q^2)^2 \tag{2}$$

Giving

$$p^4 - 2p^2q^2 + q^4 \qquad p^4 + 2p^2q^2 + q^4 \tag{3}$$

These can be equated by adding $4p^2q^2$ to the left-hand side, so that

$$(p^2 - q^2) + 4p^2q^2 = (p^2 + q^2)^2 \tag{4}$$

Noting that $4p^2q^2$ is a perfect square, we can write:

$$(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$$

which is precisely the Pythagorean relationship whereby the sum of two squares equals a third square, or:

$$a^2 + b^2 = c^2$$

where $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

Hence, Pythagorean triples can be found by taking any two positive integers p and q and calculating (a) difference of their squares, (b) twice their product, and (c) the sum of their squares.

E.g., $p = 2$, $q = 1$.

$$a = 4 - 1 = 3; b = 2(2) = 4; c = 4 + 1 = 5$$

$$3^2 + 4^2 = 5^2$$

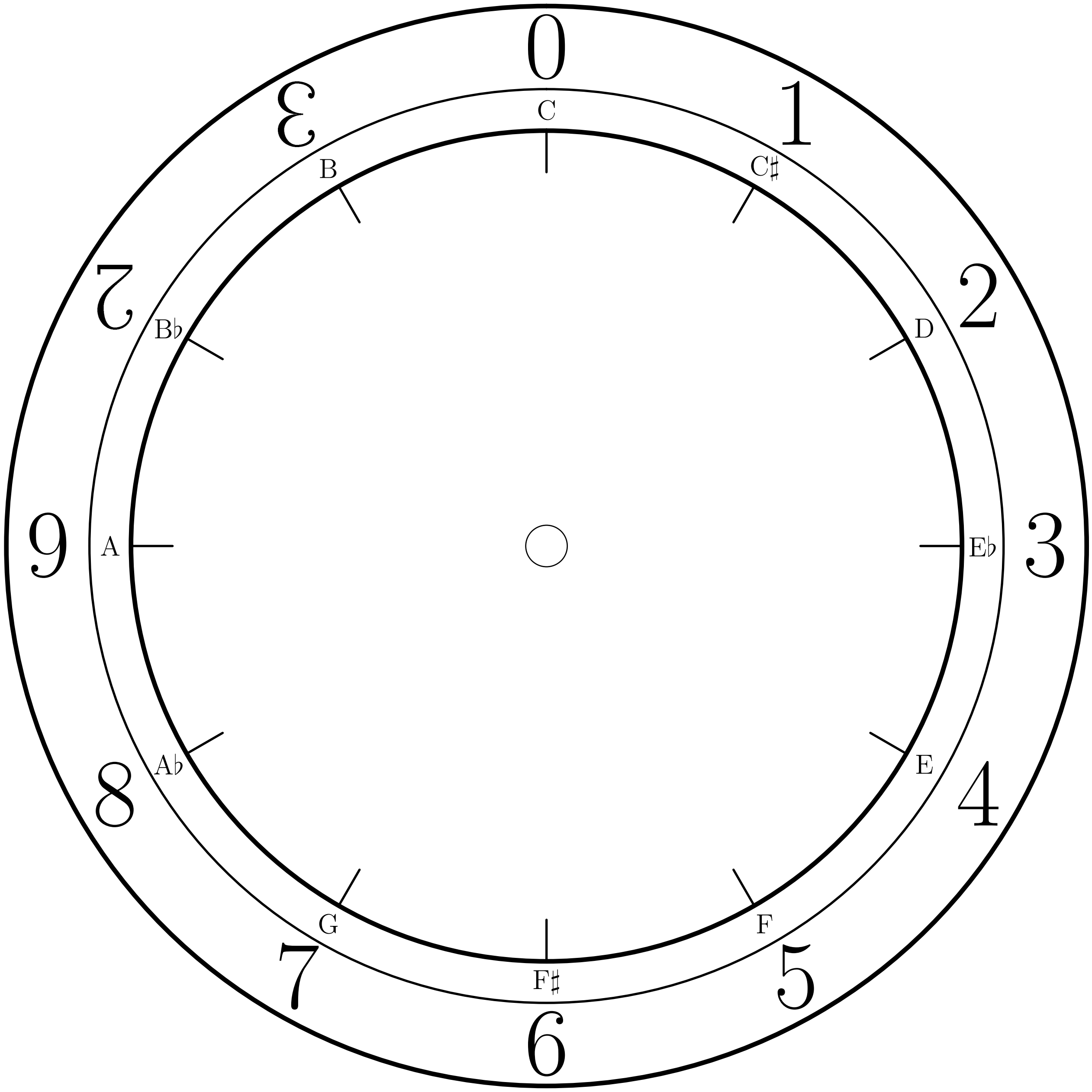
$p = 4$, $q = 1$.

$$a = 14 - 1 = 13; b = 2(4) = 8; c = 14 + 1 = 15$$

$$13^2 + 8^2 = 15^2$$

This article was original published in The Dozenal Journal, as here, under the pseudonym of "Troy"; Troy was the legendary British dozenalist Donald Hammond. That issue was published in 119ɛby the Dozenal Society of Great Britain. It is presented here as originally published, with the following exceptions: the Oxford comma has been inserted throughout; ellipses have been removed in several places where they seemed unecessary; and all the figures have been redrawn. Some additional instructions for the actual dial-a-chord device, presuming it to be made from paper, has also been added. Otherwise, it is presented as originally published, and proudly made available by the Dozenal Society of America (`http://www.dozenal.org`).

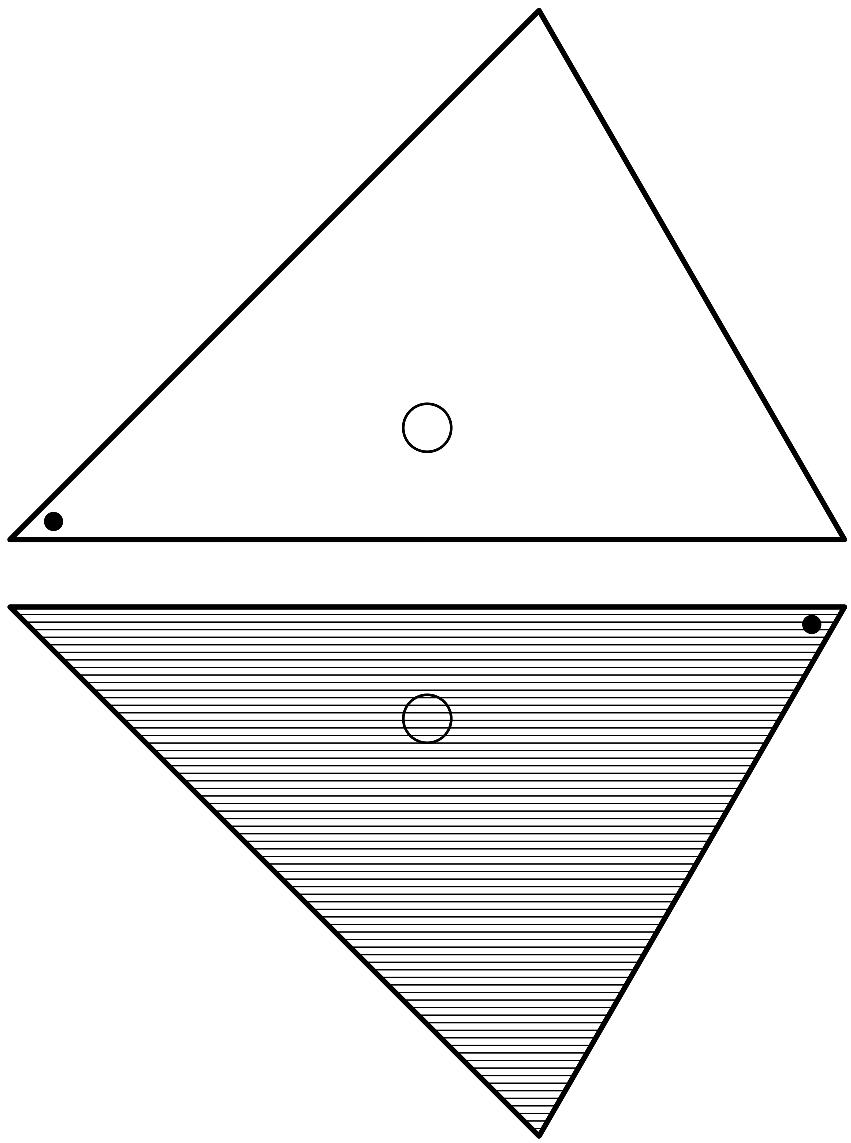
Chord Dialer



Dial-a-Chord

Major Chords Turn the white triangle so that the dotted corner points toward the primary note; the other corners will point toward the other two notes.

Minor Chords Turn the hatched triangle so that the dotted corner points toward the primary note; the other corners will point toward the other two notes.



1. Cut out the white triangle. Stay as close to the lines as possible.
2. Cut out the hatched triangle. Stay as close to the lines as possible.
3. Punch holes through the circle in the center of the “Chord Dialer” sheet, and the circles in both triangles.
4. Align the holes in the three objects in the following order: “Chord Dialer,” white triangle, hatched triangle.
5. Attach the three objects whatever way is convenient; be sure that they can rotate independently of one another.
6. Begin dialing chords.